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CHARACTERISTICS AND CONSTANTS OF MOTION METHOD FOR COLLISIONAL KINETIC EQUATIONS

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CHARACTERISTICS AND CONSTANTS OF MOTION METHOD FOR COLLISIONAL KINETIC EQUATIONS

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INTRODUCTION:

The constants of motion method* has been widely used to solve the Vlasov equation, and also for stability analysis†. For this equation the method coincides with Lagrange's characteristic method: The Lagrange characteristic differential equations are identical with the equations of motion connected with the Vlasov equation. This is, however, not necessarily the case for other kinetic equations.

In this paper we consider collisional kinetic equations and their characteristics. We also intend to use the constants of motion method to solve these equations. There are two main reasons for this investigation:

1. This method of integrating kinetic equations is able to deliver exact nonlinear solutions (being of interest for nonlinear stability analysis, for nonlinear damping, etc.).

2. Linearized solutions of the Vlasov equation breakdown when $\frac{\partial f_1}{\partial u}$ becomes the same order of magnitude as $\frac{\partial f_0}{\partial u}$. Collisional effects may be proportional to f_1 or to $\frac{\partial f_1}{\partial u}$ and should therefore be included into a nonlinear analysis (e.g. for a better matching together the linear and the nonlinear solutions, mainly for big wave lengths and low frequencies).

THE COLLISIONAL KINETIC EQUATIONS

The Lagrange characteristics of the Vlasov equation, coinciding with the respective equations of motions, read

$$\frac{d\vec{x}}{dt} = \vec{u}, \; \frac{d\vec{u}}{dt} = - \frac{e}{m} \vec{E} - \frac{e}{mc} [\vec{u} \times \vec{B}] \tag{1}$$

In order to include collisional effects (encounters) due to long-range forces, we include a Langevin term[14,15] and obtain a new equation

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* References 1 through 6

† References 7 through 13

of motion (Langevin equation)

$$\frac{d\vec{x}}{dt} = \vec{u}, \ \frac{d\vec{u}}{dt} = -\frac{e}{m}\vec{E} - \frac{e}{mc}[\vec{u} \times \vec{B}] - \nu\vec{u} \equiv \vec{b} \qquad (2)$$

where ν is an effective collision frequency. If such a dynamical friction term is included, the Hamiltionian canonical equations of motions (which are used in the derivation of the Liouville equation resp the Vlasov equation) produce a kinetic equation

$$\frac{\partial f}{\partial t} + (\vec{u}\nabla)f - (\frac{e}{m}\vec{E} + \frac{e}{mc}[\vec{u} \times \vec{B}])\nabla_u f = \nu f \qquad (3)$$

of the Bhatnagar-Gross-Krook type [16]. The Lagrange characteristics of (3) are, however, given by (1) and an additional equation

$$\frac{1}{f}\frac{df}{dt} = \nu \qquad (4)$$

We see that for the BGK equation (3) the Lagrange characteristic equations and the equations of motion are not the same.

If one asks the question for which kinetic equation the characteristics are described by the equation of motion (2) and by (4) one gets another kinetic equation

$$\frac{\partial f}{\partial t} + (\vec{u}\nabla)f + (\vec{b}\nabla_u)f = \nu f \qquad (5)$$

An even more sophisticated collisional kinetic equation was given by Chandrasekhar and others [17,18,19]

$$\frac{\partial f}{\partial t} + (\vec{u}\nabla)f + (\vec{b}\nabla_u)f = 3\nu f + q\nabla_u^2 f \qquad (6)$$

where q is a constant. All these collisional kinetic equations (3), (5) and

(6) may be derived also by various assumptions and neglections from the Fokker-Planck equation[20].

The right-hand terms of the collisional kinetic equations are also of interest for particle belt physics: now the Liouville theorem does no more forbid particle trapping since collisions within the field may change the particle properties[21]. There is experimental evidence for such a change occurring in the Van Allen belt[21] and therefore in favor of the use of collisional equations.

TRANSFORMATIONS OF THE KINETIC EQUATIONS

Since we shall show later how to solve (6), when we have a solution of (5), we first solve (5). We write

$$f(\vec{x},\vec{u},t) = e^{\nu t}\, g(\vec{x},\vec{u},t) \tag{7}$$

and obtain from (5)

$$\frac{\partial g}{\partial t} + (\vec{u}\nabla)g + (\vec{b}\nabla_u)g = 0 \tag{8}$$

which is a Vlasov equation with a Langevin collision term. The Lagrange characteristic of (8) are given by the equations of motion (2). So we may now solve the collisional kinetic equations (5), (6) and (8) and also (3) - by the constants of method.

If $a_1 \ldots a_6$ are six constants of motion of (2)

$$a_i(\vec{x},\vec{u},t) = \mathrm{const} \tag{9}$$

so that

$$\begin{aligned} \vec{u} &= \vec{u}(a_1 \ldots a_6,t) \\ \vec{x} &= \vec{x}(a_1 \ldots a_6,t) \end{aligned} \tag{10}$$

is a solution of (2), then $g(\vec{x},\vec{u},t) = g(a_1 \ldots a_6)$ is a solution of (8) since

$$\sum_{i=1}^{6} \frac{\partial g}{\partial a_i} \left(\frac{\partial a_i}{\partial t} + (\vec{u}\nabla) a_i + (\vec{b}\nabla_u) a_i \right) = \sum_i \frac{\partial g}{\partial a_i} \frac{da_i}{dt} = 0 \qquad (11)$$

Another useful transformation is the transformation to a wave frame moving with the constant velocity $\vec{v}$. The variables in this wave frame may be

$$\vec{r} = \vec{x} - \vec{v}t, \quad \vec{w} = \vec{u} - \vec{v} \qquad (12)$$

so that

$$g(\vec{x},\vec{u},t) = F(\vec{r},\vec{w}) \qquad (13)$$

presents a solution of (8). The solution F is now determined by

$$(\vec{W}\nabla_r)F + (\vec{b}\nabla_w)F = 0 \qquad (14)$$

It is time independent if the fields $\vec{E}$ and $\vec{B}$ are time independent. If one assumes[22] $F(\vec{r},\vec{w},t)$ (e.g. for special damping investigations) one obtains

$$\frac{\partial F}{\partial t} + (\vec{W}\nabla_r)F + (\vec{b}\nabla_w)F = 0 \qquad (15)$$

<u>CONSTANTS OF MOTION:</u>

In order to solve our collisional kinetic equation (8) we need constants of motion of the Langevin equations of motion. Multiplying (2) by $\vec{u}$ we obtain

$$\vec{u}\,\frac{d\vec{u}}{dt} = -\frac{e}{m}\,\vec{E}(\vec{x},t)\frac{d\vec{x}}{dt} - \nu\vec{u}\frac{d\vec{x}}{dt} \qquad (16)$$

Since $\vec{u} = \vec{u}(t)$, $\vec{x} = \vec{x}(t)$, we may write $\vec{u} = \vec{u}(x)$ and integrate

$$a(\vec{x},\vec{u},t) = \frac{u^2}{2} + \frac{e}{m}\,\phi(\vec{x},t) + \nu\psi(\vec{x}) \qquad (17)$$

where

$$\phi = \int \vec{E}(\vec{x},t)d\vec{x}, \quad \psi = \int\vec{u}(\vec{x})d\vec{x} \qquad (18)$$

The question if these integrals exist depends on the special problem. The first equation of (2) gives $(i = 1,2,3)$

$$b_i(\vec{x},\vec{u},t) = t - \int\frac{dx_i}{\sqrt{2a-u_k^2-u_l^2-2\phi-2\psi}} = t - \int\frac{dx_i}{u_i} \qquad (19)$$

where i,k,l are cyclic permutations of 1, 2, 3, $x_1 = x$, $x_2 = y$, etc. Now $g(a,b_1,b_2,b_3)$ is a particular solution of (8) which can be proven directly by insertion. When treating special problems, two further constants of motion may be found.

THE CHANDRASEKHAR EQUATION:

In treating (6) we follow ideas which were discussed by Chandrasekhar.[17,18,19] For simplification we discuss only the one-dimensional case

$$\frac{\partial f}{\partial t} + u\,\frac{\partial f}{\partial x} + b\,\frac{\partial f}{\partial u} = \nu f + q\,\frac{\partial^2 f}{\partial u^2} \qquad (20)$$

Using (7) we obtain

$$\frac{\partial g}{\partial t} + u\,\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial u} = q\,\frac{\partial^2 g}{\partial u^2} \qquad (21)$$

Introducing g(a,b,t) into (21) where a and b from (17) to (22) respectively we get

$$\frac{\partial g}{\partial t} = q\ \left(\frac{\partial^2 g}{\partial a^2} u^2 + \frac{\partial g}{\partial a}\right) \tag{23}$$

Putting $u^2(t) = G(t)$ we obtain the particular solution

$$g(a,t) = \exp[q\int G(t)dt - qt - a] \tag{24}$$

The methods proposed in this note shall be applied to forthcoming papers to collisional nonlinear electrostatic modes, to collisional nonlinear Landau damping, to collisional nonlinear damping of large amplitude whistler waves, etc.

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